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# Design of variable wavelength retarders

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**Abstract :** A retarder is a device used to change polarization states of light. It can also be used to produce phase changes in light whose state is an eigenstate of the device. Wellknown examples are quarter waveplates and half waveplates. Using these as building blocks, devices have been designed which represent rotations on the Poincare sphere about arbitrary axes through arbitrary angles (Bhandari 1989, Simon and Mukunda 1989). But these devices work as desired only at one wavelength. The purpose of the present paper is to produce a single device which works correctly at all wavelengths.

**Keywords :** Waveplates, variable wavelength retarders, Poincare sphere.

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## 1. Introduction

The Jones matrix* (Jones 1941, Jones and Hurwitz 1941) for rotation on the Poincare sphere by an angle $\alpha$ about a point on the sphere whose angular coordinates are $(\theta_0, \eta)$ is given by (Simon et *al* 1988)

$$J(\theta_0, \eta ;\ \alpha) = \begin{pmatrix} \cos \alpha/2 ; & -\cos \theta_0 \sin \alpha/2 \\ \cos \theta_0 \sin \alpha/2 ; & \cos \alpha/2 \end{pmatrix} - i \sin \alpha/2 \sin \theta_0 \begin{pmatrix} \cos \eta ; & \sin \eta \\ \sin \eta ; & -\cos \eta \end{pmatrix} \tag{0.1}$$

The sign convention used for $\alpha$ is best grasped by noting the effect on a linear polarization $\begin{pmatrix}1\\0\end{pmatrix}$ (i.e. the point on the $X$ axis.) of a Jones matrix representing a rotation about the $Y$ axis $\left(\theta_0 - \frac{\pi}{2}, \eta - \frac{\pi}{2}\right)$ through an angle $\alpha = \pi/2$.

We find that

$$\begin{pmatrix}1\\0\end{pmatrix} \rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} \tag{0.11}$$

$LP \rightarrow LHC$

*A complete set of important papers in Polarization Optics can be found in *Polarization Optics*, 1975 edited by W Swindell.

This means that anticlockwise rotations are positive.

Half waveplates are represented by rotations about points on the equator by an angle $\sphericalangle = \pi$.

We therefore have

$$H^{\circ}(\eta/2) = -i \begin{pmatrix} \cos\eta \,;& \sin\eta \\ \sin\eta \,;& -\cos\eta \end{pmatrix} \tag{0.2}$$

Physically, the angle in the parenthesis refers to the angle made by the fast axis of the waveplate with the X axis in real space.

Quarter waveplates are, similarly, rotations about points on the equator by an angle $\sphericalangle = \pi/2$

$$Q^{\circ}(\eta/2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \,;& 0 \\ 0 \,;& 1 \end{pmatrix} - \frac{i}{\sqrt{2}} \begin{pmatrix} \cos\eta \,;& \sin\eta \\ \sin\eta \,;& -\cos\eta \end{pmatrix} \tag{0.3}$$

These devices, however, produce the retardations given only at the wavelength specified by the manufacturer. At other wavelengths the corresponding representations are

$$H(\eta/2) = \cos\delta \begin{pmatrix} 1 \,;& 0 \\ 0 \,;& 1 \end{pmatrix} - i \sin\delta \begin{pmatrix} \cos\eta \,;& \sin\eta \\ \sin\eta \,;& -\cos\eta \end{pmatrix} \tag{0.4}$$

$$Q(\eta/2) = \cos\delta/2 \begin{pmatrix} 1 \,;& 0 \\ 0 \,;& 1 \end{pmatrix} - i \sin\delta/2 \begin{pmatrix} \cos\eta \,;& \sin\eta \\ \sin\eta \,;& -\cos\eta \end{pmatrix} \tag{0.5}$$

where $\delta(\neq \pi/2)$ is the new retardation.

Here we assume that the same material is used to manufacture both the waveplates. A wellknown formula (Jenkins and White) 1937 then gives

$$\delta = (2\pi/\lambda).d.(\Delta\mu), \; \delta_0 \equiv \pi/2 = (2\pi/\lambda_0)d.(\Delta\mu) \tag{0.51}$$

$$\delta = \frac{\pi}{2}\left(\frac{\lambda_0}{\lambda}\right) \tag{0.52}$$

The maximum change in $\delta$ that is possible if the wavelength is changed (we are working in the visible spectrum) is

$$|\Delta\phi| = \frac{\pi}{2}\left[\frac{\lambda_0}{\lambda^2}\,|\Delta\lambda|\right]_{\max} \tag{0.53}$$

Typically the maximum change in the retardation would be

$$\lambda \approx \lambda_0 \approx 5500\ \text{Å} \qquad \Delta\lambda \simeq 1500\ \text{Å}$$

$$|\Delta\phi| = 25^{\circ} \tag{0.54}$$

Geometrically, rotation about a point on the Poincare sphere can be synthesised by traversing a closed circuit starting from that point (Bhandari 1989). The main results of their paper are summarised below.

A VCR (Variable Circular Retarder) rotates about the polar axis. A VLR (Variable Linear Retarder) rotates about an axis on the equator. A VGER (Variable General Elliptic Retarder) rotates about an arbitrary axis.

$$\text{VCR}: H^{\circ}(\eta/2+\alpha/4)H^{\circ}(\eta/2) \tag{0.6}$$

$$\text{VLR}: Q^{\circ}(\pi/4+\eta/2)H^{\circ}(-\pi/4+\alpha/4+\eta/2)Q^{\circ}(\pi/4+\eta/2) \tag{0.7}$$

$$\text{VGER}: Q^{\circ}(\pi/2+\phi/2+\eta/2)H^{\circ}(\theta/2-\pi/4+\phi/2+\eta/2) \times Q^{\circ}(\phi/2+\eta/2)H^{\circ}(\pi/4+\eta/2) \tag{0.8}$$

with

$$\text{(i)}\quad \cos\theta = -\sin\alpha/2\,\sin\theta_0 \tag{0.81}$$

$$\text{(ii)}\quad \sin\theta\cos\phi = \sin\alpha/2\,\cos\theta_0 \tag{0.82}$$

$$\text{(iii)}\quad \sin\theta\sin\phi = -\cos\alpha/2 \tag{0.83}$$

## 2. Design of a variable wavelength linear retarder*

A linear retarder is a device that produces a rotation about a point on the equator on the Poincare sphere by an arbitrary angle.

The Jones matrix for a VLR is

$$J(\pi/2, \eta;\ \alpha) = \cos\alpha/2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\sin\alpha/2 \begin{pmatrix} \cos\eta & \sin\eta \\ \sin\eta & -\cos\eta \end{pmatrix} \tag{1.1}$$

where $\alpha$ is the retardation produced and $(\pi/2, \eta)$ is the point on the equator about which the rotation is taking place.

It is obvious that if the wavelength is changed the design given above (0.7) will not work. In this section, an elegant geometrical fact is made use of to overcome this difficulty. Consider the following question –Is it possible to draw two intersecting circles on a sphere such that equal arclengths (say $2\delta$) are enclosed in each of the circles but the intersecting area remains variable ?

That this is possible to achieve to a certain extent can be seen by performing the following experiment—Take two pieces of string each of length $2\delta$ and tie the ends together so as to form a loop. Place this loop on the surface of a sphere. Now vary the distance between the two ends and simultaneously adjust the strings so that they form arcs of circles. One can intuitively feel that the area increases as the two ends are brought closer and one can see that there a limit to which this area can be increased.

Now draw two such circles on the Poincare sphere so that the equator passes through their centres (Figure 1). This is exactly the arrangement for a VLR. A closed circuit about the point on the equator which is intersected by one of the circular arcs can be formed by purely $\delta$ and $2\delta$ retarders (i.e. quarter and

* This is abbreviated as VWVLR.

half waveplates at wavelengths other than that specified by the manufacturer). The circuit consists of

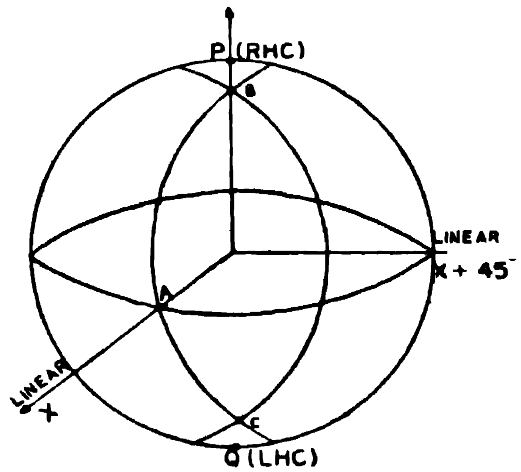


**Figure** VLR synthesis.

(i) The 'quarter' waveplate which takes an arbitrary point A to B.

(ii) The 'half' waveplate which takes B to C.

(iii) The 'quarter' waveplate which takes C back to A.

The variable wavelength VLR now becomes

$$Q(\delta/2+\eta'/2)H(-\delta/2+\beta/4+\eta'/2)Q(\delta/2+\eta'/2) \tag{1.2}$$

When explicit matrices are written down, multiplied and compared with the standard Jones matrix for a VLR one finds that the above gadget rotates about a point $(\pi/2, \eta)$ through an angle given by

$$|\sin(\alpha/4)| = |(\cos\Delta\phi)(\cos u)| \tag{1.3}$$

$$\delta = \pi/2 + \Delta\phi \tag{1.31}$$

where $u = \beta/4 - \delta$ is the angle between the plates. $\eta'$ and $\eta$ are related by

$$\begin{aligned} -\sin\alpha/2\cos\eta = & -\sin\delta\,.\,\cos\delta\,.\,\cos(\delta+\eta') \\ & -\cos^2\delta/2\,.\,\sin\delta\,.\,\cos(\eta'+\beta/2-\delta) \\ & +\sin^2\delta/2\,.\,\sin\delta\,.\,\cos(\eta'-\beta/2+3\delta) \end{aligned} \tag{1.4}$$

$$\begin{aligned} -\sin\alpha/2\sin\eta = & -\sin\delta\,.\,\cos\delta\,.\,\sin(\delta+\eta') \\ & -\cos^2\delta/2\,.\,\sin\delta\,.\,\sin(\eta'+\beta/2-\delta) \\ & +\sin^2\delta/2\,.\,\sin\delta\,.\,\sin(\eta'-\beta/2+3\delta) \end{aligned} \tag{1.5}$$

It is clear that not all values of retardation from 0 to $2\pi$ can be achieved for $\Delta\phi \neq 0$. The bounds are easily established as

$$(\sin \alpha/4)_{max} = \cos \Delta\phi \tag{1.51}$$

$$(\sin \alpha/4)_{min} = -\cos \Delta\phi \tag{1.52}$$

But as mentioned in the introduction we also have

$$|\Delta\phi| \approx 25^\circ \Rightarrow \alpha_{max} \approx 260^\circ \tag{1.53}$$

We find that even in the extreme case we can use this VLR to produce a retardation of $\pi$ or $\pi/2$. As a result exact quarter and half waveplates can be produced at any wavelength using this single gadget. The new quarter and half waveplates would then be

$$Q^\circ(\eta/2) = Q(\delta/2 + \eta_1'/2)H(-\delta/2 + \beta_1/4 + \eta_1'/2)Q(\delta/2 + \eta_1'/2) \tag{1.6}$$

$$H^\circ(\eta/2) = Q(\delta/2 + \eta_2'/2)H(-\delta/2 + \beta_2/4 + \eta_2'/2)Q(\delta/2 + \eta_2'/2) \tag{1.7}$$

with

$$\beta_1 = 4\cos^{-1}\left(\frac{\sin \pi/8}{\sin \delta}\right) + 4\delta \tag{1.80}$$

$$\beta_2 = 4\cos^{-1}\left(\frac{\sin \pi/4}{\sin \delta}\right) + 4\delta \tag{1.81}$$

$\eta_1'$ and $\eta_2'$ are related to $\eta$ by eqs. (1.4) and (1.5)

These can be used as the building blocks to produce all other kinds of retarders as described in the introduction. A summary of the results is given in the next section.

*Working of the gadgets :*

Detailed experiments using these gadgets have been carried out (Bhandari and Samuel 1988). In the experiments involving VLRs (operating at the correct wavelength) the quarter waveplates (QWPs) are held fixed and the half waveplates (HWPs) are free to rotate. The angle between the fast axis of the HWP and that of the QWP is related to the retardation $\alpha$ produced by the VLR in the manner indicated by eq. (1.3) (with $\Delta\phi = 0$). This fixes the magnitude but not the sign of the retardation produced. However, the sign is irrelevant because a different choice of the sign for $\alpha$ will yield (from eqs. (1.4) and (1.5)) an axis of rotation (defined by $\eta$) diametrically opposite to the earlier one ($\eta \rightarrow \eta - \pi$) and the two rotations are physically identical.

In our case, we are primarily interested in using the VWVLRs as QWPs or HWPs at a given wavelength. This allows us to fix the angle between the plates that comprise the VWVLR and consequently the retardation. Having done this, we can rotate the gadget as a whole to produce the retardation about any desired axis.

## 3. Unbounded variable wavelength retarders

A summary of the variable wavelength retarders (VWRs) is given below. Here $H$ and $Q$ refer to half and quarter waveplates that produce retardations of $\pi$ and $\pi/2$ respectively at a wavelength say $\lambda_0$ and are subjected to a wavelength say $\lambda$.

VCR :

$$\begin{aligned} R(\alpha) = {} & Q(\delta/2+\phi/2)H(-\delta/2+\psi/4+\phi/2) \\ & Q(\delta/2+\phi/2)Q(\delta/2+\xi/2) \\ & H(-\delta/2+\chi/4+\xi/2)Q(\delta/2+\xi/2) \end{aligned} \tag{2.1}$$

with

$$\psi=\chi=4\cos^{-1}\left(\frac{\sin \pi/4}{\sin \delta}\right)+4\delta \tag{2.11}$$

$\xi$ is found by inverting (1.4) and (1.5) and solving for $\eta'$ with $\beta=\psi$ [note that $\eta$ is arbitrary]. $\phi$ is found similarly, except that now $\eta$ is replaced by $\eta+\alpha/2$. [Here $\delta=\frac{\pi}{2}\left(\frac{\lambda_0}{\lambda}\right)$]

VLR :

$$\begin{aligned} L(\alpha\,;\ \eta) = {} & Q(\delta/2+\phi/2)H(-\delta/2+\psi/4+\phi/2)Q(\delta/2+\phi/2) \\ & Q(\delta/2+\omega/2)H(-\delta/2-\chi/4+\omega/2)Q(\delta/2+\omega/2) \\ & Q(\delta/2+\phi/2)H(-\delta/2+\psi/2+\phi/2)Q(\delta/2+\phi/2) \end{aligned} \tag{2.2}$$

with

$$\psi=4\cos^{-1}\left(\frac{\sin \pi/8}{\sin \delta}\right)+4\delta \tag{2.21}$$

$$\chi=4\cos^{-1}\left(\frac{\sin \pi/4}{\sin \delta}\right)+4\delta \tag{2.22}$$

$\phi$ is found by inverting (1.4) and (1.5) with $\beta=\psi$ and $\eta$ replaced by $\eta+\pi/2$. $\omega$ is found by inverting (1.4) and (1.5) with $\beta=\chi$ and $\eta$ replaced by $\eta+\alpha/2-\pi/2$.

In this way one can construct an unbound VGER using twelve waveplates.

## 4. Conclusions

We have succeeded in synthesizing general retarders that can produce any retardation value from 0 to $2\pi$ and most importantly the same gadget can be used at all wavelengths to get the desired effects.

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